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Extending tests for convergence of number series

E. Liflyand [a,*], S. Tikhonov [b], M. Zeltser [c]

[a] Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
[b] ICREA and Centre de Recerca Matemática (CRM), Bellaterra (Barcelona) E-08193, Spain
[c] Department of Mathematics, Tallinn University, Narva mnt. 25, 10120 Tallinn, Estonia

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ABSTRACT

Analyzing several classical tests for convergence/divergence of number series, we relax the monotonicity assumption for the sequence of terms of the series. We verify the sharpness of the obtained results on corresponding classes of sequences and functions.

1. Introduction

In various well-known tests for convergence/divergence of number series

$$\sum_{k=1}^{\infty} a_k, \tag{1.1}$$

with positive a_k, *monotonicity* of the sequence of $\{a_k\}$ is the basic assumption. Such series are frequently called monotone series. As examples, we mention tests by Abel, Cauchy, de la Vallee Poussin, Dedekind, Dirichlet, du Bois Reymond, Ermakov, Leibniz, Maclaurin, Olivier, Sapogov, Schlömilch (see, e.g., [7,3,6], or [2]); several such tests were named after Abel and Cauchy. Attempts have been made to relax the monotonicity condition for the Maclaurin–Cauchy test in older papers [14], where quasi-monotone sequences were studied, and [16], for functions of bounded variation, and in the recent paper [4].

The main goal of this paper is to continue this study and show that many of these tests are applicable not only to monotone sequences but also to those from a wider class. It is defined as follows (Leindler [8]).

Definition 1. We call a non-negative null (that is, tending to zero at infinity) sequence $\{a_k\}$ *weak monotone*, written *WMS*, if for some positive absolute constant C it satisfies

$$a_k \leqslant C a_n \quad \text{for any } k \in [n, 2n]. \tag{1.2}$$

To introduce a counterpart for functions, we will assume in this work all functions to be defined on $(0, \infty)$, locally of bounded variation, and vanishing at infinity.

* Corresponding author.
E-mail addresses: liflyand@math.biu.ac.il (E. Liflyand), stikhonov@crm.es (S. Tikhonov), mariaz@tlu.ee (M. Zeltser).

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Definition 2. We say that a non-negative function f defined on $(0,\infty)$, is *weak monotone*, written *WM*, if

$$f(t) \leqslant Cf(x) \quad \text{for any } t \in [x, 2x]. \tag{1.3}$$

Setting $a_k = f(k)$ in this case, we obtain $\{a_k\} \in WMS$.

Clearly, in these definitions $2n$ and $2x$ can be replaced by $[cn]$ (where $[a]$ denotes the integer part of a) and cx, respectively, with some $c > 1$ and another constant C.

In some problems one should consider a smaller class than *WMS*. Denote

$$\Delta a_k = a_{k+1} - a_k.$$

Definition 3. A positive null sequence $\{a_k\}$ is called *general monotone* if it satisfies

$$\sum_{k=n}^{2n} |\Delta a_k| \leqslant Ca_n, \tag{1.4}$$

for any n and some absolute constant C.

Such sequences were introduced in [15]; we shall write $\{a_k\} \in GMS$. The aforementioned class of quasi-monotone sequences [14], that is, $\{a_k\} \in QMS$ if there exists $\tau > 0$ so that $k^{-\tau} a_k \downarrow$, is a proper subclass of *GMS* [15]. It turned out that the *GMS* class is useful for many applications [9,15]. One of the simple basic properties of *GMS* is (1.2). Thus,

$$MS \subsetneq QMS \subsetneq GMS \subsetneq WMS,$$

where *MS* is the class of monotone sequences.

A similar to *GMS* function class was introduced in [9].

Definition 4. We say that a non-negative function f is general monotone, *GM*, if for all $x \in (0,\infty)$

$$\int_x^{2x} \left|df(t)\right| \leqslant Cf(x). \tag{1.5}$$

In particular, as in the case of sequences, property (1.3) is satisfied by all *GM* functions.

The reader can find a detailed survey of the properties of general monotone sequences and functions in [10]. We just remark that if $f(\cdot)$ is a *GM* function, then for $a_k = f(k)$ there holds $\{a_k\} \in GMS$. Indeed, it follows from

$$|a_k - a_{k+1}| \leqslant \left| \int_k^{k+1} df(t) \right| \leqslant \int_k^{k+1} \left|df(t)\right|.$$

The main results of the paper, that are contained in Theorems 2.1, 3.1, 3.2, and propositions from Section 5, may be summarized as the following statement.

Theorem A. *Let $f(\cdot) \in WM$. Then the following series and integrals converge or diverge simultaneously*:

$$\int_1^\infty f(t)\,dt; \qquad \sum_k f(k);$$

$$\sum_k (u_{k+1} - u_k) f(u_k), \quad \textit{provided } u_k \uparrow,\ u_{k+1} = O(u_k);$$

$$\sum_k u_k f(u_k), \qquad \sum_k u_k \left|f(u_{k+1}) - f(u_k)\right|, \quad \textit{provided } u_k \textit{ is lacunary and } u_{k+1} = O(u_k);$$

$$\sum_k k\left|f(k+1) - f(k)\right|, \qquad \int_1^\infty t\left|df(t)\right|, \quad \textit{provided } f(\cdot) \textit{ is general monotone.}$$

The first line concerns the Maclaurin–Cauchy integral test (see e.g. [6]); the second and third lines cover Cauchy's condensation test (see [2]) and its extensions [11,13]. In the last line we deal with general monotone functions/sequences [9]. Our results and examples demonstrate that *WMS* is the widest possible class, in a certain sense, of the problems here studied.

The paper is organized as follows. After Introduction, in the next section we discuss and extend the Maclaurin–Cauchy integral test and its generalization, Ermakov's test. Section 3 is devoted to Cauchy's condensation test, that is, equiconvergence of the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} \Delta u_k a_{u_k}$, and its generalizations. In Section 4 we prove several results on behavior of the terms of series with weak monotone coefficients. In Section 5, we study equiconvergence of the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} |\Delta a_{u_k}| u_k$, which is a dual version of theorems from Section 3. We assume here that $\{a_k\}$ are sequences from the General Monotone class, a subclass of *WM* sequences. In the last section we, on the one hand, show that the obtained positive results fail to hold for the classes of sequences that are natural extensions of *WMS*, and, on the other hand, present certain convergence results in which monotonicity cannot be replaced by *WM*.

By C or $C_1, C_2, \ldots,$ we denote absolute constants, that may be different in different occurrences. Notations $\lesssim$ and $\gtrsim$ mean $\leqslant C$ and $\geqslant C$, respectively, when we do not wish to indicate the constants explicitly.

2. Maclaurin–Cauchy integral test and Ermakov's test

Let us start with apparently the most applicable test, the Maclaurin–Cauchy integral test. We then deal with the closely related Ermakov's test.

2.1. Maclaurin–Cauchy integral test

In its initial form the Maclaurin–Cauchy integral test reads as follows:

Consider a non-negative monotone decreasing function f defined on $[1, \infty)$. Then the series

$$\sum_{k=1}^{\infty} f(k) \tag{2.1}$$

converges if and only if the integral

$$\int_1^{\infty} f(t)\,dt \tag{2.2}$$

is finite. In particular, if the integral diverges, then the series diverges as well.

In our extended version the *WM* property substitutes for the monotonicity.

Theorem 2.1. *Let f be a WM function. Then series* (2.1) *and integral* (2.2) *converge or diverge simultaneously.*

Proof. Applying (1.3), we have

$$\int_1^{\infty} f(t)\,dt = \sum_{k=1}^{\infty} \int_k^{k+1} f(t)\,dt \lesssim \sum_{k=1}^{\infty} f(k).$$

Further, again by (1.3)

$$\sum_{k=3}^{\infty} f(k) \lesssim \sum_{k=3}^{\infty} \int_{k/2}^{k} t^{-1} f(t)\,dt = \sum_{k=3}^{\infty} \int_k^{k+1} \int_{k/2}^{k} t^{-1} f(t)\,dt\,du \lesssim \sum_{k=3}^{\infty} \int_k^{k+1} \int_{u/3}^{u} t^{-1} f(t)\,dt\,du \lesssim \int_3^{\infty} u^{-1} \int_{u/3}^{u} f(t)\,dt\,du$$
$$\lesssim \int_1^{\infty} f(t)\,dt.$$

The proof is complete. □

2.2. Ermakov's test

In its simplest form, it is given in [7, Ch. IX, §40, 177] (or in [6]) as follows.

Let f be a continuous (*this is not necessary*) *non-negative monotone decreasing function for $t > 1$. If for t large enough $f(e^t)e^t/f(t) \leqslant q < 1$, then series* (1.1) *converges, while if $f(e^t)e^t/f(t) \geqslant 1$, then series* (1.1) *diverges.*

In fact, it is mentioned in [6] that one can get a family of tests by replacing e^t with a positive increasing function $\varphi(t)$ satisfying certain properties. We now present a generalization of the latter assertion for *WM* functions.

Theorem 2.2. *Let f be a WM function and let $\varphi(t)$ be a monotone increasing, positive function having a continuous derivative and satisfying $\varphi(t) > t$ for all t large enough.*

If for t large enough

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \leqslant q < 1, \tag{2.3}$$

then series (1.1) *converges, while if*

$$\frac{f(\varphi(t))\varphi'(t)}{f(t)} \geqslant 1, \tag{2.4}$$

then series (1.1) *diverges.*

Proof. Suppose that inequality (2.3) holds for $t \geqslant x_0$, then

$$\int_{\varphi(x_0)}^{\varphi(x)} f(t)\,dt = \int_{x_0}^{x} f\big(\varphi(u)\big)\varphi'(u)\,du \leqslant q \int_{x_0}^{x} f(t)\,dt.$$

Hence, for sufficiently large x,

$$\begin{aligned}(1-q)\int_{\varphi(x_0)}^{\varphi(x)} f(t)\,dt &\leqslant q\left[\int_{x_0}^{x} f(t)\,dt - \int_{\varphi(x_0)}^{\varphi(x)} f(t)\,dt\right] = q\left[\int_{x_0}^{\varphi(x_0)} f(t)\,dt + \int_{\varphi(x_0)}^{x} f(t)\,dt - \int_{\varphi(x_0)}^{x} f(t)\,dt - \int_{x}^{\varphi(x)} f(t)\,dt\right] \\ &\leqslant q\int_{x_0}^{\varphi(x_0)} f(t)\,dt.\end{aligned}$$

Therefore

$$\int_{\varphi(x_0)}^{\varphi(x)} f(t)\,dt \leqslant \frac{q}{1-q}\int_{x_0}^{\varphi(x_0)} f(t)\,dt.$$

Adding $\int_{x_0}^{\varphi(x_0)} f(t)\,dt$ to both sides, we obtain

$$\int_{x_0}^{\varphi(x)} f(t)\,dt \leqslant \frac{1}{1-q}\int_{x_0}^{\varphi(x_0)} f(t)\,dt = L.$$

It follows from $\varphi(x) > x$ that

$$\int_{x_0}^{x} f(t)\,dt \leqslant L,$$

for $x \geqslant x_0$, hence the integral $\int_{x_0}^{\infty} f(t)\,dt$ converges. Then Theorem 2.1 implies the desired result.

Now, we suppose that inequality (2.4) holds. Then

$$\int_{\varphi(x_0)}^{\varphi(x)} f(t)\,dt \geqslant \int_{x_0}^{x} f(t)\,dt.$$

Again, adding $\int_{x}^{\varphi(x_0)} f(t)\,dt$ to both sides, we obtain

$$\int_{x}^{\varphi(x)} f(t)\,dt \geqslant \int_{x_0}^{\varphi(x_0)} f(t)\,dt = \gamma > 0.$$

Defining the sequence $\{x_n\}$ with $x_n = \varphi(x_{n-1})$, we thus have

$$\int_{x_{n-1}}^{x_n} f(t)\,dt \geqslant \gamma.$$

Therefore

$$\int_{x_0}^{x_n} f(t)\,dt \geqslant n\gamma.$$

Hence the integral $\int_{x_0}^{\infty} f(t)\,dt$ diverges, so by Theorem 2.1, series (1.1) diverges as well. □

3. Cauchy condensation test and its generalizations

The well-known Cauchy condensation test states that

If $\{a_k\}$ is a positive monotone decreasing null sequence, then series (1.1) and the series

$$\sum_{k=1}^{\infty} 2^k a_{2^k} \tag{3.1}$$

converge or diverge simultaneously.

This assertion is a partial case of a similar extension of the following classical result due to Schlömilch (see, e.g., [7, Ch. III, §14, 77] or [2, p. 44]).

Let (1.1) be a series whose terms are positive and non-increasing, and let $u_0 < u_1 < u_2 < \cdots$ be a sequence of positive integers such that

$$\frac{\Delta u_k}{\Delta u_{k-1}} \leqslant C. \tag{3.2}$$

Then series (1.1) converges if and only if the series

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k} \tag{3.3}$$

converges.

Schlömilch's test, in turn, was generalized by de la Vallée Poussin (see [12, Th. 1 and 1a]). We now give an extension of the latter — as above it is just the same assertion but with monotonicity foregoing the *WMS* condition.

Theorem 3.1. *Let $\{u_k\}$ be an increasing sequence of positive numbers such that $u_{k+1} = O(u_k)$ and $u_k \to \infty$. Let f be a WM function. Then both series*

$$\sum_{k=1}^{\infty} f(u_k)\Delta u_k \tag{3.4}$$

and

$$\sum_{k=1}^{\infty} f(u_{k+1})\Delta u_k \tag{3.5}$$

converge (or diverge) with $\int_1^{\infty} f(t)\,dt$.

Proof. Given $k \in \mathbb{N}$, let $N_k \in \mathbb{N}$ be the number such that $2^{N_k} \leqslant u_k < 2^{N_k+1}$. Then we note that since

$$2^{N_{k+1}-N_k-1} = \frac{2^{N_{k+1}}}{2^{N_k+1}} < \frac{u_{k+1}}{u_k} \leqslant C_1,$$

it follows that $N_{k+1} - N_k \leqslant C_2$.

Given $t \in [u_p, u_{p+1}]$, let s be the integer such that $2^s \leqslant t < 2^{s+1}$. Then $s \in \{N_p, \ldots, N_{p+1}\}$. In view of (1.2) we now have

$$f(t) \leqslant Cf\left(2^s\right) \leqslant C^2 f\left(2^{s-1}\right) \leqslant \cdots \leqslant C^{s-N_p} f\left(2^{N_p+1}\right) \leqslant C^{s-N_p+1} f(u_p), \quad s > N_p.$$

Since

$$s - N_p + 1 \leqslant N_{p+1} - N_p + 1 \leqslant C_2 + 1,$$

the inequality $f(t) \leqslant C_3 f(u_p)$ follows. Then

$$C^{-1} f(u_{k+1})(u_{k+1} - u_k) \leqslant \int_{u_k}^{u_{k+1}} f(t)\,dt \leqslant C f(u_k)(u_{k+1} - u_k).$$

Therefore

$$C_5 \sum_{n=1}^{k} f(u_{n+1})(u_{n+1} - u_n) \leqslant \int_{u_1}^{u_{k+1}} f(t)\,dt \leqslant C_6 \sum_{n=1}^{k} f(u_n)(u_{n+1} - u_n). \tag{3.6}$$

Letting $k \to \infty$ presents the proof of convergence/divergence only in one direction.

To complete the proof of the theorem, it suffices to show that the convergence of the integral implies that for (3.4), and, similarly, the divergence of the integral implies that for (3.5).

For any $k \in \mathbb{N}$ we can choose $r_k \in \mathbb{N}$ such that $2^{r_k - 1} u_k \leqslant u_{k+1} \leqslant 2^{r_k} u_k$. Since $u_{k+1} = O(u_k)$, we get $r_k \leqslant r$ for some $r \in \mathbb{N}$.

Now, the divergence of $\int_{u_1}^{\infty} f(t)\,dt$ implies the divergence of

$$\sum_k f\big(2^r u_k\big) \Delta\big(2^r u_k\big) = 2^r \sum_k f\big(2^r u_k\big) \Delta u_k,$$

where we apply (3.6) to the sequence $\{2^r u_k\}$ instead of $\{u_k\}$. Since

$$f(u_{k+1}) \geqslant C^{-1} f\big(2^{r_k} u_k\big) \geqslant C^{-(r - r_k + 1)} f\big(2^r u_k\big) \geqslant C^{-r} f\big(2^r u_k\big),$$

the divergence of $\sum_k f(u_{k+1}) \Delta u_k$ follows.

Further, the convergence of $\int_{u_1}^{\infty} f(t)\,dt$ similarly implies the convergence of

$$\sum_k f\big(2^{-r} u_{k+1}\big) \Delta\big(2^{-r} u_k\big) = 2^{-r} \sum_k f\big(2^{-r} u_{k+1}\big) \Delta u_k.$$

Since $f(u_k) \leqslant C f(2^{-r} u_{k+1})$, we obtain the convergence of (3.4). The proof is complete. □

Theorem 3.2. *Let $\{a_k\}$ be a WMS, and let $u_1 \leqslant u_2 \leqslant \cdots$ be a sequence of integers such that $u_k \to \infty$ and $u_{k+1} = O(u_k)$. Then series* (1.1) *converges if and only if the series*

$$\sum_{k=1}^{\infty} \Delta u_k a_{u_k} = \sum_{k=1}^{\infty} (u_{k+1} - u_k) a_{u_k} \tag{3.7}$$

converges.

Proof. Follows from Theorem 3.1 and the Maclaurin–Cauchy test (Theorem 2.1). □

Remark 3.3. Note that the condition $\frac{\Delta u_k}{\Delta u_{k-1}} \leqslant C$ implies the condition $u_{k+1} \leqslant C u_k$, so Theorem 3.2 implies the Schlömilch test.

Indeed, it follows from (3.2) that

$$u_{k+1} - u_k \leqslant C(u_k - u_{k-1}).$$

Hence

$$\sum_{k=2}^{n} (u_{k+1} - u_k) \leqslant C \sum_{k=2}^{n} (u_k - u_{k-1}),$$

i.e., $u_{n+1} - u_2 \leqslant C(u_n - u_1)$. This yields

$$\frac{u_{n+1}}{u_n} \leqslant C + \frac{u_2 - C u_1}{u_n} \leqslant C_1.$$

Remark 3.4. Let us mention that Theorems 3.1 and 3.2 also imply the following Littlewood-type extension [11,13]: If $d_n > 0$, $D_n = \sum_{\nu=1}^{n} d_\nu$ and $f(\cdot) \in WM$, then $\sum d_n f(D_n) < \infty$ converges or diverges with $\int_1^{\infty} f(x)\,dx < \infty$, provided $D_{n+1} \leqslant C D_n$ (or $d_n \leqslant C$, or $d_{n+1} \leqslant C d_n$).

Taking in Theorem 3.2 $u_k = 2^k$, we see that $\Delta u_k = 2^k$, so we get the following extension of the classical Cauchy condensation test.

Theorem 3.5. *Let $\{a_k\}$ be a WMS. Then series* (1.1) *and* (3.1) *converge or diverge simultaneously.*

The following examples show that the condition $u_{k+1} = O(u_k)$ is essential.

Example 3.6. (1) We set $u_k = k!$ and $a_{u_k} = 1/(k!k^3)$. Then the series

$$\sum_k (u_{k+1} - u_k) a_{u_k} = \sum \frac{k!k}{k!k^3}$$

converges. Our aim is to define the remaining terms a_k so that the sequence to be *WMS* but the series $\sum_k a_k$ diverges.

Given $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}$ be such that $2^{k(n)} < n+1 \leqslant 2^{k(n)+1}$. We set

$$a_k := \begin{cases} 2^i a_{u_n} & \text{for } k = 2^{i-1}n! + 1, \dots, 2^i n!,\ i = 1, \dots, k(n), \\ 2^{k(n)+1} a_{u_n} & \text{for } k = 2^{k(n)}n! + 1, \dots, (n+1)! - 1. \end{cases}$$

Then the sequence $\{a_k\}$ is *WMS*, but the series $\sum_k a_k$ diverges, since

$$\sum_k a_k \geqslant \sum_n a_{u_n} n!\bigl(1 \cdot 2 + 2 \cdot 4 + \dots + 2^{k(n)-1} \cdot 2^{k(n)}\bigr) = \sum_n \frac{1}{2n^3} \sum_{i=1}^{k(n)} 4^i$$
$$= \sum_n \frac{1}{2n^3}\left(\frac{4^{k(n)+1} - 1}{3} - 1\right) \geqslant \sum_n \frac{1}{2n^3}\left(\frac{(n+1)^2}{3} - 1\right) = \infty.$$

(2) Now, we set $u_n = n!$ and $a_{u_n} = 1/(n!n^2)$. Then the series

$$\sum_n (u_{n+1} - u_n) a_{u_n} = \sum \frac{1}{n}$$

diverges. We will define the remaining terms a_k so that the sequence to be *WMS* with the series $\sum_k a_k$ convergent.

Given $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}$ be such that $2^{k(n)} < n \leqslant 2^{k(n)+1}$. We set

$$a_k := \begin{cases} 2^{-i} a_{u_n} & \text{for } k = 2^{-i}n!, \dots, 2^{-i+1}n! - 1,\ i = 1, \dots, k(n), \\ 2^{-k(n)-1} a_{u_n} & \text{for } k = (n-1)! + 1, \dots, 2^{-k(n)}n! - 1. \end{cases}$$

Then the sequence $\{a_k\}$ is *WMS* and the series $\sum_k a_k$ converges, since

$$\sum_k a_k \leqslant \sum_n a_{u_n} n! \sum_{i=1}^{k(n)+1} 4^{-i} + \sum_n a_{u_n} = \sum_n \frac{1}{n^2}\left(\frac{4 - 4^{-k(n)-1}}{3} - 1\right) + \sum_n a_{u_n} < \infty.$$

4. Behavior of the terms of series

In the theory of series with monotone terms there are statements not directly on convergence/divergence of the series but on the behavior of its terms. Such is Abel–Olivier's k-th term test, related to Cauchy's condensation test (see, e.g., [7, Ch. III, §14, 80]):

Let $\{a_k\}$ be a positive monotone null sequence. If series (1.1) *is convergent, then ka_k is a null sequence.*

The same result holds for a wider class *WMS*.

Theorem 4.1. *Let $\{a_k\}$ be a WMS. If series* (1.1) *converges, then ka_k is a null sequence.*

Proof. By Theorem 3.5, series (3.1) converges. Hence $2^k a_{2^k}$ is a null sequence. Given $j \in \mathbb{N}$, let k be the integer such that $2^k \leqslant j < 2^{k+1}$. Then

$$0 < ja_j < 2^{k+1} C a_{2^k} = C_1 2^k a_{2^k}.$$

Hence ja_j tends to zero, as required. □

Let a_k, $a_k \geqslant 0$, denote the k-th term of a **convergent** series and b_k, $b_k \geqslant 0$, denote the k-th term of a **divergent** series. It is worth mentioning several known facts on comparative behavior of the convergent and divergent series (see [7, Ch. IX, §41]):

(a) *Given* $\sum a_k$, *there exists a monotone* $\sum b_k$ *such that* $b_k \to 0$ *and* $\underline{\lim}\, b_k/a_k = 0$.
(b) *Given* $\sum b_k$ *such that* $b_k \to 0$, *there exists* $\sum a_k$ *such that* $\overline{\lim}\, a_k/b_k = +\infty$.

To compare the behavior of terms of weak monotone convergent and divergent series, we follow Dvoretzky [5] who investigated monotone series. Earlier results on this topic are due to Pringsheim and Hamming (see [5]).

Analyzing Dvoretzky's proof, we see that the statement can be extended like above.

Theorem 4.2. *If* $\{a_k\}, \{b_k\} \in WMS$, *and the series are convergent and divergent, respectively, then for every* $M > 1$ *there exist infinitely many* R_j, $R_j \to \infty$, *such that for all* k *with* $R_j \leqslant k \leqslant MR_j$, *we have* $a_k < b_k$.

Proof. The proof goes along the same lines as that of Theorem 1 in [5]. We first put $\Sigma_k = \{j \in \mathbb{N}\colon\ M^{k-1} \leqslant j < M^k\}$ and

$$A_k = \sum_{j\in\Sigma_k} a_j \quad \text{and} \quad B_k = \sum_{j\in\Sigma_k} b_j.$$

If $a_\nu \geqslant b_\nu$ for some $\nu \in \Sigma_k$, $k > 1$, then the *WMS* condition (rather than monotonicity in [5]) implies

$$\frac{A_{k-1}}{M^{k-1} - M^{k-2}} \geqslant C \frac{B_{k+1}}{M^{k+1} - M^k}. \tag{4.1}$$

Indeed, $a_\nu \leqslant C_1 a_l$ for $l \in \Sigma_{k-1}$. On the other hand, $b_s \leqslant C_2 b_\nu$ for $s \in \Sigma_{k+1}$. By this $b_s \leqslant C_3 a_l$, and (4.1) follows.

Since (4.1) is equivalent to $A_{k-1} \geqslant CM^{-2} B_{k+1}$, and the series $\sum A_k$ is convergent, while $\sum B_k$ is divergent, there must be infinitely many k for which the last inequality does not hold. Hence there are infinitely many k for which $a_j < b_j$ for all $j \in \Sigma_k$. This is what the theorem asserts. □

In fact, more general results can be obtained from the following simple corollary of Theorem 3.2.

Corollary 4.3. *Let* $\{a_k\}$ *be a WMS, and let* $u_1 \leqslant u_2 \leqslant \cdots$ *be a sequence of integers such that* $u_k \to \infty$ *and* $u_{k+1} = O(u_k)$. *If* $\{a_k\}, \{b_k\} \in$ *WMS and* $a_{u_n} \geqslant Cb_{u_n}$, *then the convergence of* $\sum a_n$ *implies the convergence of* $\sum b_n$.

We then generalize Theorem 4.2 by taking $u_k = M^k$.

5. General monotone sequences and functions

Our next result is a dual result of Schlömilch-type test (Theorem 3.2). We recall that the increasing sequence $\{u_k\}$ is called *lacunary* if $u_{k+1}/u_k \geqslant q > 1$. A more general class of sequences is that in which each sequence can be split into finitely-many lacunary sequences (see, e.g., [1, Intr.]). In the latter case we will write $\{u_k\} \in \Lambda$. This is true if and only if

$$\sum_{j=1}^{k} u_j \leqslant C u_k. \tag{5.1}$$

In different terms, $\{u_k\} \in \Lambda$ if and only if there exists $r \in \mathbb{N}$ such that

$$\frac{u_{k+r}}{u_k} \geqslant q > 1, \quad k \in \mathbb{N}. \tag{5.2}$$

We denote $\bar{\Delta} a_{u_k} := a_{u_k} - a_{u_{k+1}}$.

Proposition 5.1. *Let* $\{a_k\}$ *be a non-negative WMS, and let a sequence* $\{u_k\}$ *be such that* $\{u_k\} \in \Lambda$ *and* $u_{k+1} = O(u_k)$.

Then series (1.1), *and the series*

$$\sum_{k=1}^{\infty} u_k |\bar{\Delta} a_{u_k}|, \tag{5.3}$$

and

$$\sum_{k=1}^{\infty} u_k a_{u_k} \tag{5.4}$$

converge or diverge simultaneously.

Proof. First, we show that series (1.1) and (5.4) are equiconvergent. Representing

$$\sum_k a_k = \sum_k \sum_{j=u_k}^{u_{k+1}-1} a_j$$

and using $\{a_k\} \in WMS$ and $u_{k+1} = O(u_k)$, we get

$$(u_{k+1} - u_k)a_{u_{k+1}} \lesssim \sum_{j=u_k}^{u_{k+1}-1} a_j \lesssim (u_{k+1} - u_k)a_{u_k}.$$

In one direction, the statement readily follows from the last upper estimate and $(u_{k+1} - u_k) \leqslant u_{k+1} \leqslant Cu_k$.

Let $\{u_k\} \in \Lambda$, or equivalently, condition (5.2) holds. If $r = 1$, that is, the sequence $\{u_k\}$ is merely lacunary, we immediately have

$$(u_{k+1} - u_k) \geqslant (q-1)u_k \geqslant C(q-1)u_{k+1}.$$

More difficult is the lower estimate when $r > 1$. We wish to have $\sum_k \Delta u_k a_{u_{k+1}} \geqslant C \sum_k u_{k+1} a_{u_{k+1}}$. In general, this cannot be done termwise, since u_{k+1} and u_k may be close enough, and only $\Delta u_k a_{u_{k+1}} \geqslant C a_{u_{k+1}}$ is possible. However, let us prove that

$$\sum_k \Delta u_k a_{u_{k+1}} \geqslant C \sum_k u_{l_k+1} a_{u_{l_k+1}},$$

where $l_k \leqslant k$, with $k - l_k \leqslant r$ for each k. Indeed, it follows from (5.2) that $\frac{u_{k+1}}{u_{k+1-r}} \geqslant q > 1$ for every k. Also, there exists $\{l_k\}$ such that $0 \leqslant k - l_k \leqslant r$ and $u_{l_k+1} - u_{l_k} \geqslant (u_{k+1} - u_{k+1-r})/r$. For this l_k we have

$$(u_{l_k+1} - u_{l_k})a_{u_{l_k+1}} \geqslant a_{u_{l_k+1}}(u_{k+1} - u_{k+1-r})/r \geqslant Cu_{k+1-r}a_{u_{l_k+1}}.$$

Now, the assumptions of the theorem ($\{a_k\} \in WMS$ and $u_{k+1} = O(u_k)$) give

$$u_{k+1-r}a_{u_{l_k+1}} \gtrsim u_{l_k+1}a_{u_{l_k+1}} \gtrsim u_{k+1}a_{u_{l_k+1}} \gtrsim u_{k+1}a_{u_{k+1}}.$$

Hence,

$$\sum_k u_{k+1}a_{u_{k+1}} \lesssim \sum_k \Delta u_{l_k} a_{u_{l_k+1}} \lesssim \sum_k \Delta u_k a_{u_{k+1}} \lesssim \sum_k a_k \lesssim \sum_k u_k a_{u_k},$$

and series (1.1) and (5.4) converge or diverge simultaneously.

Let us study equiconvergence of series (5.3) and (5.4). Applying Abel-type transformation, we get

$$\sum_{k=1}^N u_k a_{u_k} = \sum_{k=1}^{N-1}\left(\sum_{j=1}^k u_j\right)\bar{\Delta} a_{u_k} + \left(\sum_{j=1}^N u_j\right)a_{u_N} \leqslant C\left(\sum_{k=1}^{N-1} u_k|\bar{\Delta} a_{u_k}| + u_N \sum_{k=N}^{\infty} |\bar{\Delta} a_{u_k}|\right) \leqslant C\sum_{k=1}^{\infty} u_k|\bar{\Delta} a_{u_k}|,$$

where we have used the assumption that u_k satisfies (5.1).

On the other hand, since $\{a_k\} \in WMS$,

$$|\bar{\Delta} a_{u_k}| \leqslant a_{u_k} + a_{u_{k+1}} \leqslant C a_{u_k},$$

and we have

$$\sum_{k=1}^{\infty} u_k|\bar{\Delta} a_{u_k}| \leqslant C\sum_{k=1}^{\infty} u_k a_{u_k}.$$

The proof is complete. □

Note that we do not need lacunarity to show that series (1.1) and (5.3) converge under assumption that (5.4) converges. The following example demonstrates that both conditions, lacunarity and *WMS*, are essential in the proposition.

Example 5.2. (1) Consider $u_k = k$ which is not lacunary and $a_k = k^{-2}$, say. Then series (1.1) and (5.3) converge but the series (5.4) diverges.

(2) Let again $u_k = k$ and take a weak monotone sequence

$$a_k = \begin{cases} 2 \cdot 4^{-l}, & 2^l \leqslant k < 2^{l+1},\ k \text{ is even}; \\ 4^{-l}, & 2^l \leqslant k < 2^{l+1},\ k \text{ is odd}. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} a_k \leqslant 2 \sum_{l=1}^{\infty} 2^l 4^{-l} < \infty,$$

but

$$\sum_{k=1}^{\infty} u_k a_{u_k} \geqslant \sum_{k=1}^{\infty} u_k |\Delta a_{u_k}| \geqslant \sum_{l=1}^{\infty} 2^l \sum_{k=2^l+1}^{2^{l+1}-2} |\Delta a_k| = \sum_{l=1}^{\infty} \frac{2^l}{4^l} \sum_{k=2^l+1}^{2^{l+1}-2} 1 = \infty.$$

(3) Now, for the lacunary sequence $u_k = 2^k$ and the sequence $a_k \notin WM$ defined by

$$a_k = \begin{cases} 4^{-n}, & k = 2^n; \\ k^{-1}, & k \notin \{2^n \mid n \in \mathbb{N}\}; \end{cases}$$

series (5.3) and (5.4) converge but series (1.1) diverges.

(4) On the other hand taking the lacunary sequence $u_k = 2^k$ and the sequence $a_k \notin WM$ defined by

$$a_k = \begin{cases} 2^{-n}, & k = 2^n; \\ 2^{-k}, & k \notin \{2^n \mid n \in \mathbb{N}\}, \end{cases}$$

we get that series (5.3) and (5.4) diverge while series (1.1) converges.

However, it is also possible to get equiconvergence results for series (1.1) and (5.3) for an important case $u_n = n$. Since lacunarity can no more help, we proceed to a smaller class than *WMS*. Indeed, assuming general monotonicity of the sequences, we prove the following result.

Proposition 5.3. *Let $\{a_k\}$ be a GMS. Then series* (1.1) *and*

$$\sum_k k|\Delta a_k| \tag{5.5}$$

converge or diverge simultaneously.

Proof. On the one hand, since $\{a_k\} \in GMS$,

$$\sum_{k=1}^{\infty} a_k \gtrsim \sum_{k=1}^{\infty} \sum_{j=k}^{2k} |\Delta a_j| \simeq \sum_{k=1}^{\infty} k|\Delta a_k|.$$

On the other hand, since

$$m a_m \leqslant m \sum_{k=m}^{\infty} |\Delta a_k| \leqslant \sum_{k=1}^{\infty} k|\Delta a_k|,$$

we have

$$\left| \sum_{k=1}^{m} a_k \right| = \left| \sum_{k=1}^{m-1} (a_k - a_{k+1})k + m a_m \right| \lesssim \sum_{k=1}^{\infty} k|\Delta a_k|.$$

The proof is complete. □

We observe that Example 5.2(2) delivers a sequence $\{a_k\} \in WMS$ but not *GMS* for which

$$\sum_{k=1}^{\infty} a_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} k|\Delta a_k| = \infty.$$

Let us prove a result for functions similar to Proposition 5.3.

Proposition 5.4. *Let f be a GM function. Then the integrals $\int_1^\infty f(t)\,dt$ and $\int_1^\infty t\,|df(t)|$ converge or diverge simultaneously.*

Proof. The proof goes along the same lines as that of Proposition 5.3. Indeed, it follows from the definition of GM that

$$\int_1^\infty f(t)\,dt \gtrsim \int_1^\infty \int_t^{2t} \left|df(s)\right| dt \gtrsim \int_1^\infty t^{-1} \int_t^{2t} s\left|df(s)\right| dt \gtrsim \int_1^\infty s\left|df(s)\right|.$$

To prove reverse, since

$$Nf(N) \leqslant N \int_N^\infty \left|df(s)\right| \leqslant \int_N^\infty s\left|df(s)\right|,$$

we obtain

$$\left|\int_1^N f(t)\,dt\right| = \left|Nf(N) - f(1) - \int_1^N t\,df(t)\right| \lesssim f(1) + \int_1^\infty t\left|df(t)\right|,$$

which completes the proof. □

6. Negative type results

We now consider a group of tests for the inextensibility of monotonicity to weak monotonicity.

6.1. Sapogov's test

This interesting test reads as follows [6, Ch. 2, Problem 8] (the "divergence" part can also be found in [3, p. 36]).

If $\{b_k\}$ is a positive monotone increasing sequence, then the series

$$\sum_{k=1}^\infty \left(1 - \frac{b_k}{b_{k+1}}\right) \tag{6.1}$$

as well as

$$\sum_{k=1}^\infty \left(\frac{b_{k+1}}{b_k} - 1\right) \tag{6.2}$$

converges if the sequence $\{b_k\}$ is bounded and diverges otherwise.

This cannot be true if b_k (as well as $1/b_k$) is *WMS*. Taking $b_k = 1$, $k \neq 2^m$ and $b_k = 2$, $k = 2^m$, gives a bounded sequence is then bounded, but the series diverges. Evidently, the necessary condition for $\sum_{k=1}^\infty (1 - \frac{b_k}{b_{k+1}})$ is $\frac{b_k}{b_{k+1}} \to 1$ which is not satisfied in this example.

Sapogov's test and Theorem 3.1 are related to certain extent. Indeed, taking $f(t) = 1/t$ in Theorem 3.1 addresses the divergence part of both tests. However, the convergence part of Theorem 3.1 is strongly based on the increasing of $\{u_k\}$ to infinity, for example in (3.6).

Let us now figure out whether the monotonicity can in principle be relaxed in Sapogov's test. We say that a sequence $\{a_k\}$ is of bounded variation if

$$\sum_{k=1}^\infty |\Delta a_k| < \infty, \tag{6.3}$$

written $\{a_k\} \in BV$. The "positive" part of Sapogov's test can be generalized for BV-sequences as follows. If we suppose that $0 < C_1 < b_k$ for every k, then

$$\sum_{k=1}^\infty \left|1 - \frac{b_k}{b_{k+1}}\right| = \sum_{k=1}^\infty \frac{|b_{k+1} - b_k|}{b_{k+1}} \leqslant \sum_{k=1}^\infty \frac{|b_{k+1} - b_k|}{C_1} < \infty$$

for any sequence of bounded variation, i.e., series (6.1) and (6.2) converge absolutely.

However, there exists a sequence $\{b_k\}$ which is not of bounded variation such that

$$\sum_{k=1}^\infty \left|1 - \frac{b_k}{b_{k+1}}\right|$$

converges. We set

$$b_k = \begin{cases} 1 + 1/n, & k = 3n - 2 \text{ or } k = 3n; \\ 1 + 2/n, & k = 3n - 1. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} |b_{k+1} - b_k| \geqslant \sum_{n=1}^{\infty} \frac{C}{n} = \infty,$$

but

$$\begin{aligned} \sum_{k=1}^{\infty} \left| 1 - \frac{b_k}{b_{k+1}} \right| &= \sum_{n=1}^{\infty} \left| \frac{b_{3n-1} - b_{3n-2}}{b_{3n-1}} + \frac{b_{3n} - b_{3n-1}}{b_{3n}} + \frac{b_{3n+1} - b_{3n}}{b_{3n+1}} \right| = \sum_{n=1}^{\infty} \left| \frac{\frac{1}{n}}{1 + \frac{2}{n}} - \frac{\frac{1}{n}}{1 + \frac{1}{n}} + \frac{\frac{1}{n+1} - \frac{1}{n}}{1 + \frac{1}{n+1}} \right| \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} + \frac{1}{n(n+2)} \right) \end{aligned}$$

converges.

6.2. Tests of Dedekind, du Bois Reymond, Dirichlet, Abel, and Leibniz

The tests of Dedekind and of du Bois Reymond are combined in [7, Ch. X, §43, 184] as the next assertion.

Let $\{a_k\}$ and $\{b_k\}$ be two sequences.

(i) *If $\{a_k\} \in BV$, $\{a_k\}$ is a null sequence, and the sequence of partial sums of $\sum_k b_k$ is bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.*
(ii) *If $\{a_k\} \in BV$ and $\sum_k b_k$ is convergent, then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.*

Two well-known and widely used corollaries of this test — Dirichlet's and Abel's tests — involve monotone sequences. The first one is as follows.

Let $\{a_k\}$ be a monotone null sequence and $\{b_k\}$ be a sequence such that the sequence of its partial sums is bounded. Then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

One of corollaries of this test is the celebrated Leibniz test:

Let $\{a_k\}$ be a monotone null sequence. Then the series $\sum_{k=1}^{\infty} (-1)^k a_k$ is convergent.

Further, Abel's test reads as follows.

Let $\{a_k\}$ be a bounded monotone sequence and $\sum_k b_k$ a convergent series. Then the series $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

A counterexample can be given against extension of Abel's test. Let $a_n = 1$ everywhere except 2^k-th place where $a_{2^k} = 2$. Taking $b_n = (-1)^n / \ln n$, we get that $\sum_k (-1)^k / \ln k$ converges. However, $\sum_k a_k b_k$ is the sum of 2 series: convergent $\sum_k (-1)^k / \ln k$ and divergent $\sum_k 1/\ln 2^k = (\ln 2)^{-1} \sum 1/k$.

As for the Leibniz test, it cannot hold without additional assumption of the boundedness of variation: just take $a_k = 1/\ln k$ everywhere except $n = 2^k$ where $a_n = 2/\ln n$.

6.3. Wider classes

We have considered a series of results where extension of monotonicity to the weak one was successful. A natural question arises whether these results are sharp. We shall show that *WMS* is, in a sense, the widest class for which such tests are still valid.

An immediate natural extension of *WMS* is the class defined by

$$a_k \leqslant C \sum_{n=[k/2]}^{k} \frac{a_n}{n}, \tag{6.4}$$

or a bit more general

$$a_k \leqslant C \sum_{n=[k/c]}^{[ck]} \frac{a_n}{n} \tag{6.5}$$

for some $c > 1$.

The principle difference between these classes and *WMS* is that (6.4) and (6.5) allow certain amount of zero members, unlike *WMS* that forbid even a single zero, i.e., $a_{n_0} = 0$ implies $a_n = 0$ for $n \geqslant n_0$.

Putting zeros on certain positions, say $k = 2^n$, we easily construct a counterexample to show that the Cauchy condensation test cannot be valid for (6.4) and (6.5) as well as its extensions in Section 3 and dual results in Section 5.

In a similar way, just letting a function f to take non-zero values only close to integer points, one sees that the Maclaurin–Cauchy integral test may fail as well.

In conclusion, note that *WMS* is a subclass of the broadly used Δ_2-class, that is, the one for which the doubling condition $a_{2k} \leqslant Ca_k$ holds for each $k \in \mathbb{N}$. We observe that assuming the doubling condition by no means can guarantee the above tests to be extended. To illustrate this, let us restrict ourselves to Cauchy's condensation test. Taking $\{a_k\}$ such that

$$a_k = \begin{cases} 2^{-k}, & k \neq 2^n; \\ n^{-2}, & k = 2^n. \end{cases}$$

This sequence is doubling but not *WMS*. Obviously,

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

diverges, the desired counterexample.

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